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Fully developed free convection in open-ended vertical concentric porous annuli

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Abstract—Analytical solutions for fully developed natural convection in open-ended vertical concentric porous annuli are presented. Four fundamental boundary conditions have been investigated and the corresponding fundamental solutions are obtained. These four fundamental boundary conditions are obtained by combining each of the two conditions of having one boundary maintained at uniform heat flux or at uniform wall temperature with each of the conditions that the opposite boundary is kept isothermal at the inlet fluid temperature, or adiabatic. Expressions for the flow and heat transfer parameters are given for each case. These fundamental solutions may be used to obtain solutions satisfying more general thermal boundary conditions.

INTRODUCTION

An understanding of convective heat transfer in porous annuli is essential for its numerous applications in packed-bed catalytic reactors, geophysics, thermal insulation, design of regenerative heat exchangers, geological disposal of high-level nuclear waste, petroleum resources, and many other uses.

Free and mixed convection problems in a vertical porous annulus has been extensively studied by Prasad and Kulacki [1], Prasad *et al.* [2], Clarksean *et al.* [3], Reda [4], Muralidhar [5] and Choi and Kulacki [6, 7]. The above studies use numerical or experimental techniques to investigate the thermal behaviour of the porous annulus. Analytical solutions for the problem of natural convection in porous annulus is only possible when the flow is fully developed. Fully developed free convection flows are obtained when the inertia forces vanish and a balance is attained between the pressure, gravitational and Darcian forces on the one hand and the viscous forces on the other hand. The non-Darcian inertial effects which account for the additional pressure drop resulting from inter-pore mixing may be neglected for pure free convection situations. The study of fully developed flows gives the limiting conditions for developing flows and provides an analytical check on numerical solutions. In addition, fully developed behaviour is attained in a very short distance from the entrance of porous channels.

Nevertheless, to the author's knowledge, only one analytical study [8] is available in the literature dealing with fully developed free convection flows in vertical annuli. Parang and Keyhani [8] have obtained closed-form solutions for the special case of fully developed flow, where the inner and outer walls are heated by uniform but unequal heat fluxes. Solutions for the same case in geometries other than the annular one may be found in the literature [9]. The limiting case of fully developed natural convection in non-porous annuli, where the porosity approaches zero, is solved analytically for steady and transient cases by El-Shaarawi and Al-Nimr [10] and Al-Nimr [11].

The lack of analytical solutions for fully developed laminar natural convection in vertical concentric porous annuli, with different fundamental combinations of isothermal and isoflux thermal boundary conditions, motivated the present work. The purpose of this paper is to present, in closed forms, fully developed free convection solutions, corresponding to four fundamental thermal boundary conditions, in vertical concentric porous annuli.

GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

Consider steady fully developed free convection flow inside an open-ended vertical concentric porous annulus of a finite length (l), immersed in a stagnant fluid of infinite extent maintained at a constant temperature t_0. Figure 1 shows the physical situation in which at least one of the channel walls is heated or cooled either isothermally or at a constant wall heat flux, so that its temperature (i.e. temperature of the inner surface of the outer cylinder or that of the outer surface of the inner cylinder) is different from the ambient temperature t_0. Due to fully developed flow assumptions, the fluid enters the part under consideration of the porous annular passage with an axial velocity profile which remains invariant in the entire channel (i.e. $\partial u/\partial z = 0$). The fluid is assumed to be Newtonian, it enters the channel at the ambient temperature t_0, and both the fluid and solid matrix are assumed to be in thermal equilibrium and to have constant physical properties. Also, the fluid obeys the

NOMENCLATURE

a local heat transfer coefficient based on the area of heat transfer surface $q/(t_w - t_0) = \pm k_e(\partial t/\partial r)_w/(t_w - t_0)$, minus and plus signs apply, respectively, for heating and cooling at the inner boundary and vice versa at the outer boundary

$\bar{a}$ average heat transfer coefficient over the annulus height, based on the average temperature

A_i constants of integration, where $i = 1$ and 2, of the heat transfer boundary, $\int_0^l a dz/l$

b annular gap width, $r_2 - r_1$

B_i constants of integration, where $i = 1$ and 2

c_p specific heat of fluid at constant pressure

C_i constants of integration where $i = 1$, 2, 3 and 4

d_i constants related to C_i where $i = 1, 2, 3$ and 4

Da Darcy number, $K/\varepsilon r_2^2$

D equivalent (hydraulic) diameter of annulus, $2b$

D_w diameter of heat transfer boundary

E_i constants where $i = 1$ and 2

f volumetric flow rate, $\int_{r_1}^{r_2} 2\pi r u \, dr$

F dimensionless volumetric flow rate, $f/(\pi l \gamma Gr^*)$

g gravitational body force per unit mass

Gr Grashof number, $\mp g\beta(t_w - t_0)D^3/\gamma^2$ in the case of an isothermal boundary, or $\mp g\beta q D^4/2k_e\gamma^2$ in the case of uniform heat flux (UHF) heat transfer boundary, the plus and minus signs apply to upward (heating) and downward (cooling) flows, respectively. Thus Gr is a positive number in both cases

Gr^* modified Grashof number, $D\,Gr/l$

I_0 modified Bessel function of the first kind of order zero

I_1 modified Bessel function of the first kind of order one

k_e effective thermal conductivity of porous medium

K permeability of the porous medium

K_0 modified Bessel function of the second kind of order zero

K_1 modified Bessel function of the second kind of order one

l height of annulus

L dimensionless height of annulus, $1/Gr^*$

N annulus radius ratio, r_1/r_2

Nu local Nusselt number, $|a|D/k_e$

$\overline{Nu}$ average Nusselt number, $\int_0^l Nu \, dz/l$

p pressure of fluid inside the channel at any cross-section

p' pressure defect at any point, $p - p_s$

p_0 pressure of fluid at the channel entrance

p_s hydrostatic pressure, $\rho_0 g z$

P dimensionless pressure defect at any point, $p' r_2^4/\rho_0 l^2 \gamma^2 Gr^{*2}$

Pr_e effective Prandtl number, γ/α_e

q heat flux at the heat transfer surface, $q = \mp k_e(\partial t/\partial r)_w$, where the minus and plus signs are, respectively, for heating and cooling in case I; these signs should be reversed in case O

r radial coordinate

r_1 inner radius of annulus

r_2 outer radius of annulus

R dimensionless radial coordinate, r/r_2

t temperature at any point

t_m mixing cup temperature over any cross section, $\int_{r_1}^{r_2} rut \, dr/\int_{r_1}^{r_2} ru \, dr$

t_0 temperature at the annulus entrance

t_w temperature of heat transfer boundary

T dimensionless temperature, $(t - t_0)/(t_w - t_0)$ in the case of an isothermal heat transfer boundary, and $(t - t_0)/(qD/2k_e)$ for UHF boundary, and thus it is positive for both heating (upward) and cooling (downward) flows

T_m dimensionless mixing cup temperature, $(t_m - t_0)/(t_w - t_0)$ in the case of an isothermal heat transfer boundary, and $(t_m - t_0)/(qD/2k_e)$ for UHF boundary

u volume-averaged axial velocity

u_0 volume-averaged axial velocity at the entrance of the annulus

U dimensionless volume averaged axial velocity, $ur_2^2/(l\gamma Gr^*)$

z axial coordinate

Z dimensionless axial coordinate, $z/(lGr^*)$.

Greek symbols

α parameter defined in equation (5)

α_e effective thermal diffusivity, $k_e/\varepsilon\rho_0 c_p$

β volumetric coefficient of thermal expansion

γ kinematic viscosity of fluid, μ/ρ_0

ε porosity of the medium

θ_w dimensionless temperature of heat transfer boundary

λ parameter defined in equation (9)

μ dynamic viscosity of fluid

ρ fluid density at temperature t, $\rho_0(1 - \beta(T - T_0))$

ρ_0 fluid density at t_0.

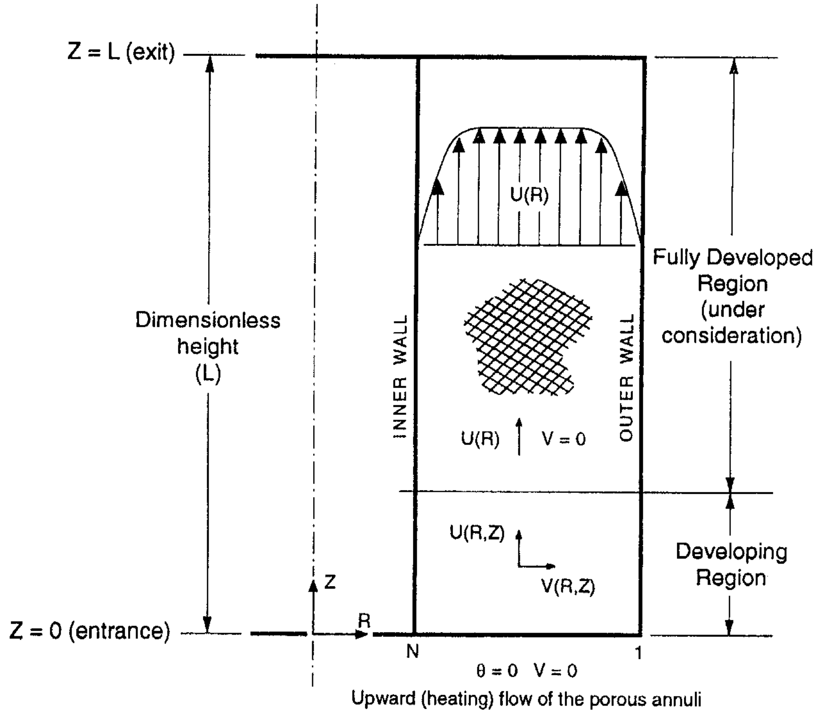

Fig. 1. Schematic diagram.

Boussinesq approximation according to which its density is constant, except in the gravitational term of the vertical momentum equation. It is assumed that the flow has axial symmetry and it is assumed that both viscous dissipation and internal heat generation are absent.

Under the above-mentioned assumptions and using the dimensionless parameters given in the Nomenclature, the equations of continuity, motion and energy reduce to the following two simultaneous non-dimensional equations:

$$-\frac{\mathrm{d}P}{\mathrm{d}Z}+\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}\left[R\frac{\mathrm{d}U(R)}{\mathrm{d}R}\right]-\frac{1}{Da}U(R)+\frac{T(R,Z)}{16(1-N)^4}=0 \quad (1)$$

$$U(R)Pr_e\frac{\partial T}{\partial Z}=\frac{1}{R}\frac{\partial}{\partial R}\left[R\frac{\partial T}{\partial R}\right]. \quad (2)$$

Four boundary conditions are therefore needed to obtain a solution for the above two second-order differential equations. The two conditions related to U are

$$U(1)=U(N)=0. \quad (3)$$

On the other hand, there are many possible thermal boundary conditions applicable to the annular configuration. In the present paper, the non-dimensional parameters used in the formulation of the problem are chosen to suit annuli having their two boundaries at two different heat fluxes (q_1 and q_2), or at two different uniform temperatures (t_1 and t_2), or annuli under one of four fundamental boundary conditions. These four fundamental boundary conditions are obtained by combining each of the two conditions of having one boundary maintained at uniform wall temperature or at specified heat flux, with each of the conditions that the opposite boundary is kept isothermal at the inlet fluid temperature (t_0) or adiabatic $\partial t/\partial r = 0$.

With the two boundaries of an annulus maintained at UHF (uniform heat flux) conditions, if q_1 refers to the larger heat flux, then q_1 will be at the hotter wall in case of heating and at the cooler wall in case of cooling. Thus, the value of r_q (ratio of the heat fluxes at the two boundaries, q_2/q_1) may vary from -1 to 1. Similarly, when the two boundaries of an annulus are kept isothermal, t_1 refers to the wall which has the larger temperature difference from t_0. Thus, t_1 is the temperature of the hotter wall in case of heating the two boundaries and of the cooler wall in case of cooling both boundaries. Therefore, the wall temperature difference ratio r_T $[=(t_2-t_0)/(t_1-t_0)]$ may, in this case of UWT boundary conditions, also vary from -1 to 1.

From the previous discussion it may be seen that there are many thermal boundary conditions applicable to the annulus geometry. However, under certain conditions, the energy equation (2) becomes linear and homogeneous in T (e.g. when $\partial T/\partial Z$ is constant), and then any linear combination of solutions will be a solution. It may then be possible to develop certain special or fundamental solutions to this equation satisfying particular or specific boundary conditions, which can be combined to satisfy any other boundary conditions. This method is known as the method of superposition. Reynolds *et al.* [12] defined four fundamental boundary conditions for the annular geometry, which produce four fundamental solutions to the energy equation (2) when it becomes linear. For the sake of completeness, these fundamental solutions are stated hereinafter.

(1) Fundamental solutions of the first kind, which satisfy the boundary conditions of a temperature step change at one wall, the opposite wall being kept isothermal at the inlet fluid temperature. Using the present notation, this corresponds to $T = 1$ at one wall and $T = 0$ at the opposite wall (i.e. $r_T = 0$).

(2) Fundamental solutions of the second kind, which satisfy the boundary conditions of a step change in heat flux at one wall, the opposite wall being adiabatic. Using the present notation, this corresponds to $\partial T/\partial R = -1/(1-N)$ at the inner wall and $\partial T/\partial R = 0$ at the outer wall, or $\partial T/\partial R = 0$ at the inner wall and $\partial T/\partial R = 1/(1-N)$ at the outer wall.

(3) Fundamental solutions of the third kind which satisfy the boundary conditions of a temperature step change, at one wall, the opposite wall being adiabatic. This corresponds to $T = 1$ at one wall and $\partial T/\partial R = 0$ at the opposite wall.

(4) Fundamental solutions of the fourth kind, where a step change in heat flux at one wall is applied, while the opposite wall is kept isothermal at the inlet fluid temperature. This corresponds to $\partial T/\partial R = -1/(1-N)$ at the inner wall, while $T = 0$ at the outer wall or $T = 0$ at the inner wall, and $\partial T/\partial R = 1/(1-N)$ at the outer wall.

With any of the above-mentioned boundary conditions, the boundary opposite to that maintained adiabatic (i.e. $\partial T/\partial R = 0$) or at t_0 (i.e. $T = 0$) is termed the heat-transfer boundary (even though there is transfer of heat through a boundary maintained at $T = 0$). For each of the above fundamental solutions, two cases are considered, namely, case I, in which the heat transfer boundary is at the inner wall, and case O, in which the heat transfer boundary is at the outer wall. The aim of the present paper is to obtain the above-mentioned four fundamental solutions.

GENERAL ANALYSIS

Substituting T from equation (1) into equation (2), we obtain

$$Pr_e U\frac{d^2P}{dZ^2} + \frac{d^4U}{dR^4} + \frac{2}{R}\frac{d^3U}{dR^3} - \left(\frac{1}{Da} + \frac{1}{R^2}\right)\frac{d^2U}{dR^2} + \left(\frac{1}{R^3} - \frac{1}{DaR}\right)\frac{dU}{dR} = 0. \quad (4)$$

A solution of equation (4) in the form $U = U(R)$ is only possible if

$$\frac{d^2P}{dZ^2} = \alpha \quad (5)$$

where α is constant. From equation (1) one may obtain

$$\frac{\partial T}{\partial Z} = 16\alpha(1-N)^4 \quad (6)$$

which means that, for a given R in a given annulus, the dimensionless temperature T varies linearly with the axial distance Z. This implies that the assumption of a hydrodynamically fully developed free convection flow should necessarily mean that the flow is also thermally fully developed, regardless of the value of the effective Prandtl number (Pr_e). In other words, for free convection flows in the vertical annulus, the thermal development length is shorter than or at most equal to that of the hydrodynamic development length, irrespective of the value of the effective Prandtl number. However, in pure forced convection flows, such a result is only obtained if $Pr_e \leqslant 1$.

Integrating equation (5) twice and applying the conditions that $P = 0$ at both inlet and exit (i.e. at $Z = 0$ and L), gives

$$P = 0.5\alpha Z(Z-L). \quad (7)$$

Substituting equation (5) into equation (4) yields

$$\frac{d^4U}{dR^4} + \frac{2}{R}\frac{d^3U}{dR^3} - \left(\frac{1}{Da} + \frac{1}{R^2}\right)\frac{d^2U}{dR^2} + \left(\frac{1}{R^3} - \frac{1}{DaR}\right)\frac{dU}{dR} + \lambda^4 U = 0 \quad (8)$$

where

$$\lambda^4 = \alpha Pr_e. \quad (9)$$

Substituting P from equation (7) into equation (1) gives

$$\alpha(Z-0.5L) - \frac{1}{R}\frac{d}{dR}\left[R\frac{dU(R)}{dR}\right] + \frac{1}{Da}U(R) = \frac{T(R,Z)}{16(1-N)^4}. \quad (10)$$

The governing equations (8)–(10) can be simplified if one of the two annulus boundaries is kept isothermal. In order to satisfy this boundary condition, T must, in this particular case, be independent of Z. Thus, it is concluded that α (and hence λ) must, in such a case, equal zero. Therefore, equations (5)–(8) and (10) reduce, in this case, to the following equations, respectively:

$$\alpha = \frac{d^2P}{dZ^2} = 0 \quad (11)$$

$$\frac{\partial T}{\partial Z} = 0 \quad (12)$$

$$P = 0 \quad (13)$$

$$\frac{d^4U}{dR^4} + \frac{2}{R}\frac{d^3U}{dR^3} - \left(\frac{1}{Da} + \frac{1}{R^2}\right)\frac{d^2U}{dR^2} + \left(\frac{1}{R^3} - \frac{1}{DaR}\right)\frac{dU}{dR} = 0 \quad (14)$$

$$T = \frac{-16(1-N)^4}{R}\frac{d}{dR}\left[R\frac{dU(R)}{dR}\right] + \frac{16(1-N)^4}{Da}U(R). \quad (15)$$

Equation (12) states that, in a case with an isothermal boundary, the fully developed temperature profile is constant or at most a function of the radial coordinate only. On the other hand, equation (13) states that the fully developed pressure inside an open-ended annulus of an isothermal boundary is equal to the hydrostatic pressure, at the same elevation, outside the annulus. This implies that, in such a fully developed case with an isothermal boundary, there would be no pressure drop due to fluid viscous or Darcian drag, since these are just offset by the buoyancy driving force.

If the two governing equations (14) and (15) or their general forms (8) and (10) are solved for the velocity and temperature profiles (U and T), then the following useful parameters can be evaluated.

The dimensionless volumetric flow rate (F) can be evaluated from the following equation:

$$F = 2\int_N^1 RU\,dR. \quad (16)$$

Since for a fully developed flow U is a function of R only, it follows that the definite integral on the right-hand side of equation (16) and hence F are constants, regardless of the value of the axial coordinate Z, i.e. they are not related to the value of the annulus height. It can, however, be shown that, in cases with two UHF boundaries, there exists a relation between this constant fully developed value of F and the thermal boundary conditions applied at the boundaries of an annulus. Integrating equation (2) with respect to R from $R = N$ to 1, the following equation is obtained:

$$\left(R\frac{\partial T}{\partial R}\right)_{R=1} - \left(R\frac{\partial T}{\partial R}\right)_{R=N} = Pr_e\int_N^1 RU\frac{\partial T}{\partial Z}dR. \quad (17)$$

However, for a given annulus, equation (6) shows that, in cases with two UHF boundaries, $\partial T/\partial Z$ is constant, and hence it can be taken out of the above integral. Substituting for $\partial T/\partial Z$ from equation (6) in equation (17) and using the result in equation (16) gives

$$F = \frac{\left[\left(\frac{\partial T}{\partial R}\right)_{R=1} - N\left(\frac{\partial T}{\partial R}\right)_{R=N}\right]}{8\lambda^4(1-N)^4}. \quad (18)$$

It may also be worth mentioning that, in a case with a UWT heat transfer boundary, equations (12) and (17) give the following result:

$$\left(\frac{\partial T}{\partial R}\right)_{R=1} = N\left(\frac{\partial T}{\partial R}\right)_{R=N}. \quad (19)$$

Take into consideration that the rate of heat transfer per unit length from the inner and outer surfaces of an annulus are given, respectively, by

$$Q_{li} = \mp 2\pi r_1 k_e\left(\frac{\partial t}{\partial r}\right)_{r=r_1} \quad (20)$$

$$Q_{lo} = \pm 2\pi r_2 k_e\left(\frac{\partial t}{\partial r}\right)_{r=r_2}. \quad (21)$$

Then, for a fundamental solution of the first kind (i.e. a case with two UWT boundaries), one may obtain

$$Q_{li} = \mp 2\pi k_e(t_w - t_0)N\left(\frac{\partial T}{\partial R}\right)_{R=N} \quad (22)$$

$$Q_{lo} = \pm 2\pi k_e(t_w - t_0)\left(\frac{\partial T}{\partial R}\right)_{R=1}. \quad (23)$$

The upper and lower (plus or minus) signs in the above expressions apply, respectively, for heating and cooling. Equations (19), (22) and (23) yield the following conclusions. In an annulus with two UWT boundaries, or an annulus with a UWT boundary and an opposite UHF boundary, when fully developed conditions are achieved, the rate of heat transfer from one boundary should be equal and opposite to that from the other boundary (i.e. $A_1q_1 = -A_2q_2$). This implies that, in such cases, the net rate of heat transfer to/from the fully developed fluid flow is zero. Thus, it is anticipated, in such cases, that the bulk fluid temperature would remain constant. However, in the special case with a UWT boundary ($T = 1$) and an opposite adiabatic boundary $\partial T/\partial R = 0$, equation (19) shows that $\partial T/\partial R$ at the UWT boundary must also vanish. Thus, in this special case, fully developed conditions are achieved when both $\partial T/\partial R$ and $\partial T/Z$ vanish, i.e. the temperature becomes uniform at the UWT boundary.

Equation (18) confirms that the fully developed dimensionless volumetric flow rate is independent of the dimensionless channel height (L) and it depends on the thermal boundary conditions applied at the two annulus boundaries. This means that, when the channel becomes sufficiently high so that the flow reaches its state of full development, a further increase in the channel height would not produce any further increase in the sucked volumetric flow rate. When fully developed conditions are achieved, in a case with two UHF boundaries, an increase in the value of F

may be obtained by increasing the heat flux at the boundaries rather than the channel height L.

The dimensionless inlet velocity U_0 is given in terms of F by

$$U_0 = \frac{F}{1-N^2}. \quad (24)$$

Therefore, U_0 is similarly constant irrespective of the annulus height and is related, in cases with UHF boundaries, to the thermal boundary conditions by the following equation:

$$U_0 = \frac{\left[\left(\frac{\partial T}{\partial R}\right)_{R=1} - \left(N\frac{\partial T}{\partial R}\right)_{R=N}\right]}{8\lambda^4(1-N)^5(1+N)}. \quad (25)$$

The dimensionless mixing cup temperature is given by

$$T_m = \frac{\int_N^1 RUT\,\mathrm{d}R}{\int_N^1 RU\,\mathrm{d}R}. \quad (26)$$

To find the variation of T_m, in the fully developed flow region, with the dimensionless axial distance Z, the above equation is differentiated with respect to Z. Since U is independent of Z, this gives

$$\frac{\partial T_m}{\partial Z} = \frac{\int_N^1 RU\frac{\partial T}{\partial Z}\mathrm{d}R}{\int_N^1 RU\,\mathrm{d}R} \quad (27)$$

which, on substituting for $\partial T/\partial Z$ from equation (6) into the above equation, yields

$$\frac{\partial T_m}{\partial Z} = 16\alpha(1-N)^4. \quad (28)$$

Integrating equation (28) with respect to Z between annulus entrance and exit, taking into consideration that $T_m = 0$ at $Z = 0$, results in

$$T_m = 16\alpha(1-N)^4 Z. \quad (29)$$

Using the dimensionless parameters given in the Nomenclature, the following expressions for the local Nusselt number can easily be obtained: for a UWT boundary

$$Nu = \pm 2(1-N)\left(\frac{\partial T}{\partial R}\right)_w \quad (30)$$

and for UHF boundaries

$$Nu = \pm\frac{2(1-N)}{T_w}\left(\frac{\partial T}{\partial R}\right)_w = \frac{2}{T_w} \quad (31)$$

where the minus and plus signs apply respectively for cases I and O when there is heating, and vice versa when there is cooling.

From equation (15) it can be seen that $(\partial T/\partial R)$ is a function of R only, which is dependent on the fully developed axial velocity profile (U), i.e. it is independent of Z. Hence, for a case with a UWT boundary, equations (15) and (30) show that the fully developed local Nusselt number is constant. Consequently, the fully developed average Nusselt number is, in this case (UWT), independent of annulus height L. On the other hand, with the two boundaries at UHF, equation (10) shows that the temperature varies linearly with Z. Hence, equations (10) and (31) show that the fully developed local Nusselt number and consequently the average Nusselt number, for a given annulus with UHF boundaries, vary hyperbolically with Z. These conclusions are as expected since, as was previously mentioned, the assumption of hydrodynamically fully developed flows implies also thermally fully developed free convection flows.

FUNDAMENTAL SOLUTIONS OF THE FIRST KIND

In this case, the two boundaries of the annulus are kept isothermal, one of which is at the inlet ambient fluid temperature t_0, while the opposite boundary is at a higher or a lower temperature.

Equation (2) (with $\partial T/\partial z = 0$) is readily solved to yield

$$T = A_1 \ln R + A_2, \quad (32)$$

where A_1 and A_2 are arbitrary constants. Substituting for T from equation (32) into equation (15), solve to yield

$$U = B_1 I_0(Da^{-1/2}R) + B_2 K_0(Da^{-1/2}R) + C_1 \ln R + C_2 \quad (33)$$

where

$$C_1 = \frac{A_1 Da}{16(1-N)^4} \quad C_2 = \frac{A_2 Da}{16(1-N)^4}.$$

The constants B_1 and B_2 are given in terms of the no slip conditions (3) as

$$B_1 = \frac{[-C_2 K_0(Da^{-1/2}N) + (C_2 + C_1 \ln N)K_0(Da^{-1/2})]}{[I_0(Da^{-1/2})K_0(Da^{-1/2}N) - K_0(Da^{-1/2})I_0(Da^{-1/2}N)]}$$

$$B_2 = \frac{[C_2 I_0(Da^{-1/2}N) - (C_2 + C_1 \ln N)I_0(Da^{-1/2})]}{[I_0(Da^{-1/2})K_0(Da^{-1/2}N) - K_0(Da^{-1/2})I_0(Da^{-1/2}N)]}.$$

To obtain the constants A_1 and A_2, the following thermal boundary conditions should be applied to equation (32):

case I—temperature step at the inner wall, while the outer wall is kept at ambient temperature, i.e.

$$T(N, Z) = 1 \quad T(1, Z) = 0.$$

case O—temperature step at the outer wall while the inner wall is kept at ambient temperature, i.e.

$$T(N,Z) = 0 \quad T(1,Z) = 1.$$

For problems of the first kind, the constants A_1 and A_2 are given as:

case I

$$A_1 = \frac{1}{\ln N} \quad A_2 = 0$$

case O

$$A_1 = \frac{-1}{\ln N} \quad A_2 = 1.$$

The volume flow rate F and the mixing cup temperature T_m, defined by equations (16) and (26), are readily calculated using solutions obtained for U and T. The results are as follows:

$$F = 2\Big[Da^{1/2}B_1RI_1(Da^{-1/2}R) - Da^{1/2}B_2RK_1(Da^{-1/2}R) + \frac{C_1}{2}R^2(\ln R) - \frac{C_1}{4}R^2 + \frac{C_2}{2}R^2\Big]_N^1 \quad (34)$$

$$T_m = \frac{2}{F}\Big[Da^{1/2}A_1B_1R(\ln R)I_1(Da^{-1/2}R) - DaA_1B_1I_0(Da^{-1/2}R) - Da^{1/2}A_1B_2R(\ln R)K_1(Da^{-1/2}R) - DaA_1B_2K_0(Da^{-1/2}R) + \frac{A_1C_1}{2}R^2(\ln^2 R) - \frac{A_1C_1}{2}R^2(\ln R) + \frac{A_1C_1}{4}R^2 + (A_1C_2 + A_2C_1)R^2\left(\frac{(\ln R)}{2} - \frac{1}{4}\right) + Da^{1/2}A_2B_1RI_1(Da^{-1/2}R) - Da^{1/2}A_2B_2RK_1(Da^{-1/2}R)\Big]_N^1. \quad (35)$$

Expressions for the fully developed local Nusselt number is obtained after getting the temperature gradient at the walls from equation (32) and substituting in equation (30). The results are:

case I

$$Nu_I = -\frac{2(1-N)}{N\ln N}$$

case O

$$Nu_O = -\frac{2(1-N)}{\ln N}.$$

FUNDAMENTAL SOLUTIONS OF THE SECOND KIND

In this case, one of the annulus boundaries is maintained at a constant heat flux (q) and the opposite boundary is perfectly insulated. The governing equations in such a case are equations (8) and (10). Equation (8) has the following solution in terms of the modified Bessel functions of zero order:

$$U = C_1I_0(\beta_1R) + C_2K_0(\beta_1R) + C_3I_0(\beta_2R) + C_4K_0(\beta_2R) \quad (36)$$

where

$$\beta_1 = \sqrt{[0.5Da^{-1} + 0.5\sqrt{(Da^{-2} - 4\lambda^4)}]}$$

$$\beta_2 = \sqrt{[0.5Da^{-1} - 0.5\sqrt{(Da^{-2} - 4\lambda^4)}]}.$$

Substitution of U from equation (36) into equation (10) yields the following solution for the temperature profile:

$$T = 16\alpha(1-N)^4(Z-0.5L) + E_1C_1I_0(\beta_1R) + E_1C_2K_0(\beta_1R) + E_2C_3I_0(\beta_2R) + E_2C_4K_0(\beta_2R) \quad (37)$$

where

$$E_1 = 16(1-N)^4(Da^{-1} - \beta_1^2)$$

$$E_2 = 16(1-N)^4(Da^{-1} - \beta_2^2).$$

The constants C_1, C_2, C_3 and C_4 are evaluated in terms of the no-slip conditions (3) and the following fundamental thermal boundary conditions:

case I—step change in heat flux at the inner wall while the outer wall is adiabatic, i.e.

$$\left.\frac{\partial T}{\partial R}\right|_{R=N} = -\frac{1}{1-N} \quad \left.\frac{\partial T}{\partial R}\right|_{R=1} = 0$$

case O—step change in heat flux at the outer wall while the inner wall is adiabatic, i.e.

$$\left.\frac{\partial T}{\partial R}\right|_{R=1} = \frac{1}{1-N} \quad \left.\frac{\partial T}{\partial R}\right|_{R=N} = 0.$$

In terms of the hydrodynamic and thermal boundary conditions, the following four equations in C_1, C_2, C_3 and C_4 result:

case I

$$\begin{bmatrix} I_0(\beta_1N) & K_0(\beta_1N) & I_0(\beta_2N) & K_0(\beta_2N) \\ I_0(\beta_1) & K_0(\beta_1) & I_0(\beta_2) & K_0(\beta_2) \\ d_1I_1(\beta_1N) & d_2K_1(\beta_1N) & d_3I_1(\beta_2N) & d_4K_1(\beta_2N) \\ d_1I_1(\beta_1) & d_2K_1(\beta_1) & d_3I_1(\beta_2) & d_4K_1(\beta_2) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dfrac{-1}{1-N} \\ 0 \end{bmatrix}$$

case O

$$\begin{bmatrix} I_0(\beta_1 N) & K_0(\beta_1 N) & I_0(\beta_2 N) & K_0(\beta_2 N) \\ I_0(\beta_1) & K_0(\beta_1) & I_0(\beta_2) & K_0(\beta_2) \\ d_1 I_1(\beta_1 N) & d_2 K_1(\beta_1 N) & d_3 I_1(\beta_2 N) & d_4 K_1(\beta_2 N) \\ d_1 I_1(\beta_1) & d_2 K_1(\beta_1) & d_3 I_1(\beta_2) & d_4 K_1(\beta_2) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{1-N} \end{bmatrix}$$

where

$$d_1 = \beta_1 E_1 \quad d_2 = -d_1 \quad d_3 = \beta_2 E_2 \quad d_4 = -d_3.$$

Each of the above two sets of equations can be solved by some standard procedure, such as Cramer's rule, to obtain the Cs in terms of βs (or λ). The value of λ can be determined by the following procedure. By definition, the dimensionless mixing cup temperature, in the present case, is given by the equation

$$T_m = \frac{2D_w Z}{Pr_e DF}. \tag{38}$$

Equating the right-hand sides of equations (29) and (38), the following expression for λ is obtained

$$\lambda^4 = \alpha Pr_e = \frac{D_w}{8DF(1-N)^4}. \tag{39}$$

This expression reduce in case I to

$$\lambda^4 = \frac{N}{8F(1-N)^5} \tag{40}$$

and in case O to

$$\lambda^4 = \frac{1}{8F(1-N)^5}. \tag{41}$$

The dimensionless volumetric flow rate is given in terms of equations (18) and (37) as

$$\begin{aligned} F = [&C_1 d_1 I_1(\beta_1) + C_2 d_2 K_1(\beta_1) \\ &+ C_3 d_3 I_1(\beta_2) + C_4 d_4 K_1(\beta_2) - NC_1 d_1 I_1(\beta_1 N) \\ &- NC_2 d_2 K_1(\beta_1 N) - NC_3 d_3 I_1(\beta_2 N) \\ &- NC_4 d_4 K_1(\beta_2 N)]/[8\alpha Pr_e(1-N)^4]. \end{aligned} \tag{42}$$

To obtain λ and the values of C_1, C_2, C_3 and C_4 a simple iterative procedure may be used. An assumed initial value of λ is used to obtain an initial set of values for Cs. These values are then used in equation (42) to obtain a value for F, and hence a second iterate for λ may be obtained from equation (40) or (41). The procedure is repeated until convergence within a specified tolerance is obtained.

Having obtained the value of λ (and hence α), equation (29) [or equation (38)] can be used to obtain T_m.

Finally, the following expressions for the fully developed local Nusselt number are obtained after substituting the values of T_w from equation (37) in equation (31):

case I

$$Nu_I = \frac{2}{[16\alpha(1-N)^4(Z-0.5L) + E_1 C_1 I_0(\beta_1 N) + E_1 C_2 K_0(\beta_1 N) + E_2 C_3 I_0(\beta_2 N) + E_2 C_4 K_0(\beta_2 N)]}$$

case O

$$Nu_O = \frac{2}{[16\alpha(1-N)^4(Z-0.5L) + E_1 C_1 I_0(\beta_1) + E_1 C_2 K_0(\beta_1) + E_2 C_3 I_0(\beta_2) + E_2 C_4 K_0(\beta_2)]}.$$

FUNDAMENTAL SOLUTIONS OF THE THIRD KIND

The two thermal boundaries associated with this fundamental case are given as:

case I—temperature step change at the inner wall while the outer wall is kept insulated, i.e.

$$\frac{\partial T}{\partial R}(1, Z) = 0 \quad T(N, Z) = 1$$

case O—temperature step at the outer wall while the inner wall is kept insulated, i.e.

$$\frac{\partial T}{\partial R}(N, Z) = 0 \quad T(1, Z) = 1.$$

The governing equations for this kind are similar to that of the first kind. As a result, the temperature, velocity, volumetric flow rate and mixing cup temperature are given as in equations (32)–(35), respectively, but with a new set of constants given for both cases I and O as

$$A_1 = 0 \quad A_2 = 1.$$

It is worth mentioning that, since the wall opposite to the heat transfer surface is perfectly insulated, the fluid temperature in the annular space becomes ultimately uniform at the same temperature as the heated surface ($T = 1$). Thus, in this case, an isothermal flow (at the temperature of the heat transfer boundary), in which a balance is attained among the buoyancy, Darcian and viscous forces, is achieved. Also, since the temperature is uniform everywhere ($T = 1$), the mixing cup temperature described by equation (35) will give $T_m = 1$ with the new set of constants. Finally, and since the full development conditions yield an isothermal flow, the fully developed local Nusselt number is zero.

FUNDAMENTAL SOLUTIONS OF THE FOURTH KIND

In this case, since one of the boundaries is isothermal, the governing equations are similar to

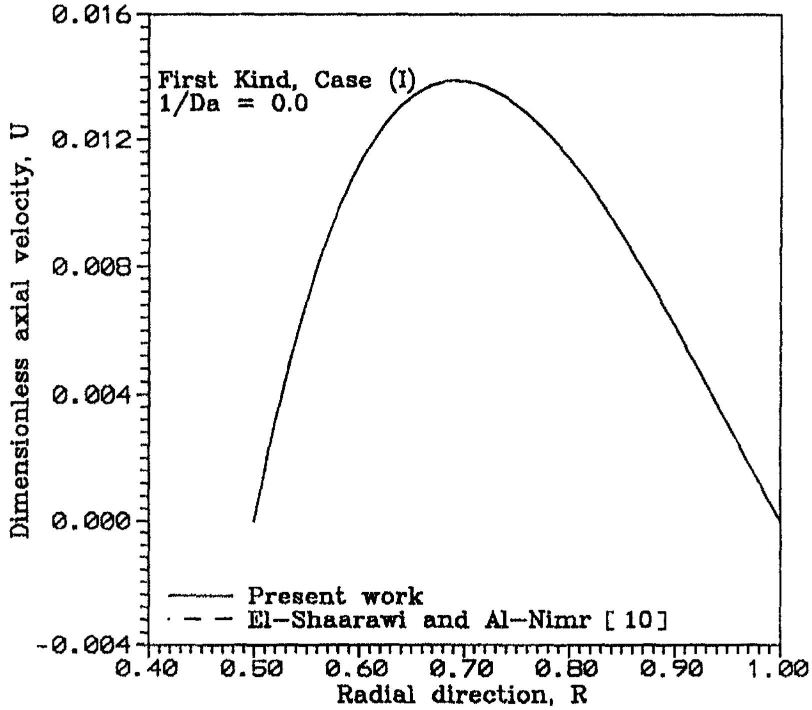

Fig. 2. Comparison of the present velocity profiles with those of El-Shaarawi and Al-Nimr [10].

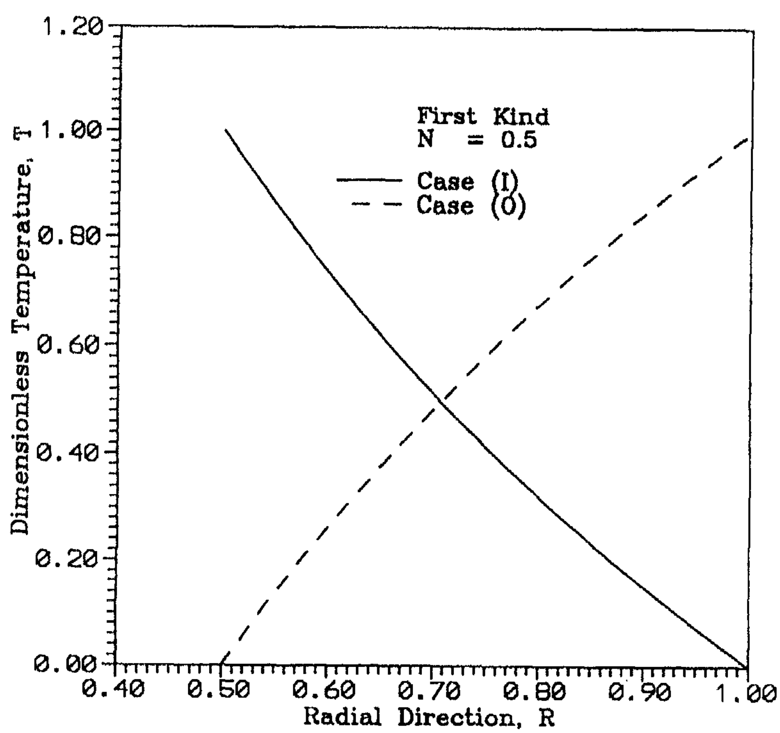

Fig. 3. Dimensionless temperature distribution in the radial direction.

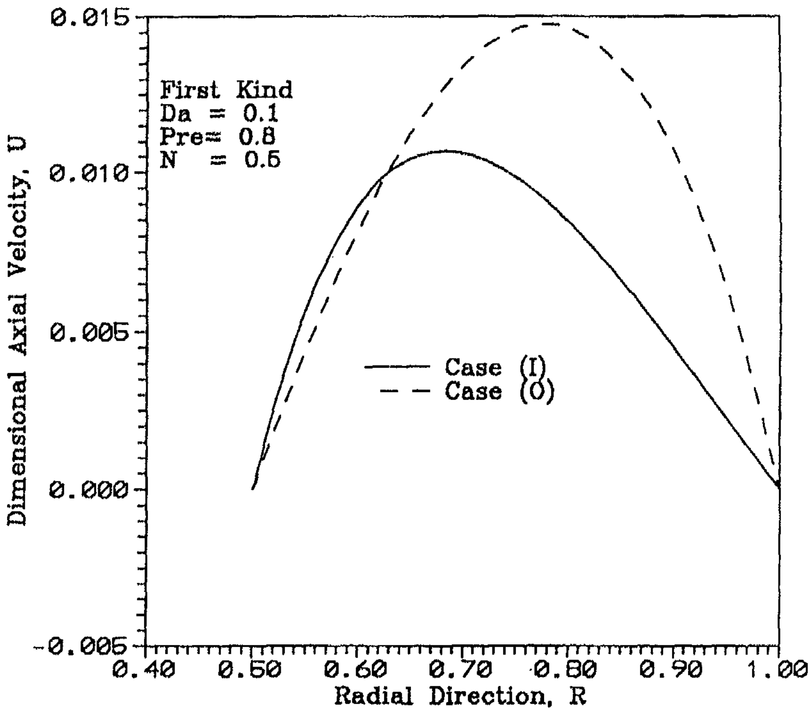

Fig. 4. Dimensionless axial velocity distribution in the radial direction.

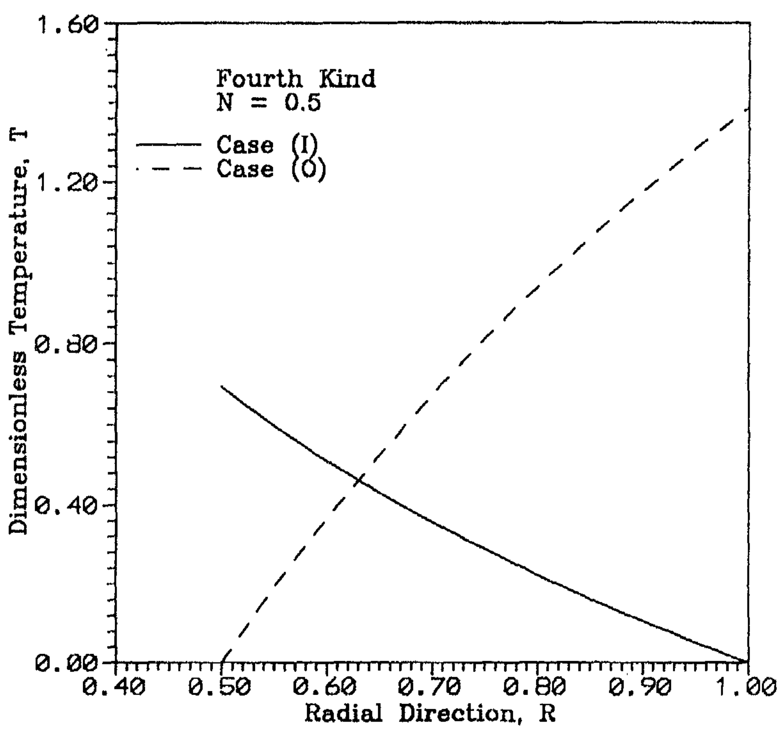

Fig. 5. Dimensionless temperature distribution in the radial direction.

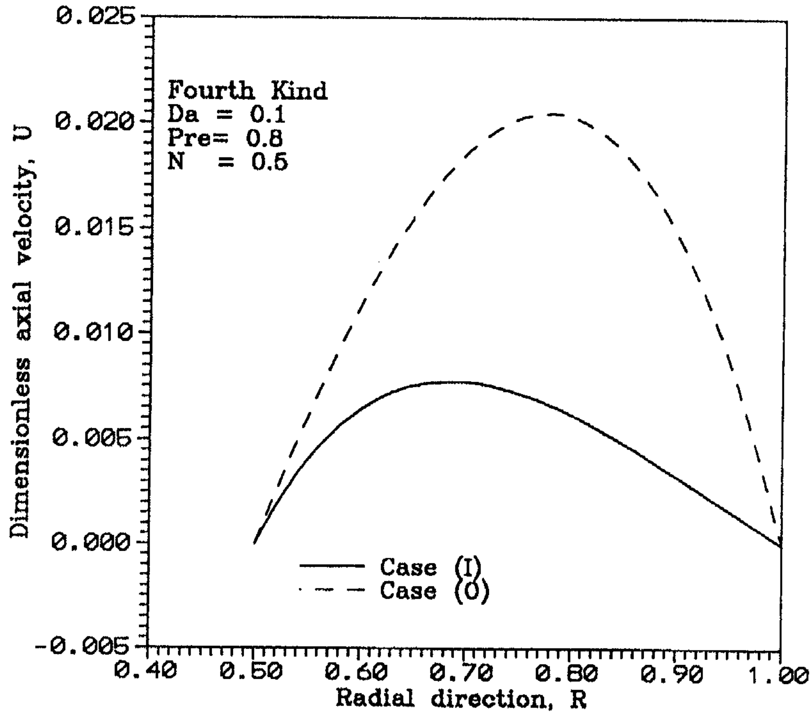

Fig. 6. Dimensionless axial velocity distribution in the radial direction.

those of the first and third kinds, but with different thermal boundary conditions given as:

case I—step change in heat flux at the inner wall while the outer wall is isothermal at the inlet fluid temperature, i.e.

$$\left.\frac{\partial T}{\partial R}\right|_{R=N} = -1/(1-N) \quad T(1,Z) = 0$$

case O—step change in heat flux at the outer wall while the inner wall is isothermal at the inlet fluid temperature, i.e.

$$\left.\frac{\partial T}{\partial R}\right|_{R=1} = 1/(1-N) \quad T(N,Z) = 0.$$

Expressions for the temperature, velocity, volumetric flow rate and mixing cup temperature are given as in equations (32)–(35), respectively, but with a new set of constants given for both cases I and O as:

case I

$$A_1 = \frac{-N}{1-N} \quad A_2 = 0$$

case O

$$A_1 = \frac{1}{1-N} \quad A_2 = \frac{-\ln N}{1-N}.$$

Expressions for the fully developed Nusselt number are obtained after getting the temperature gradient at the walls from equation (32) (but with the new sets of constants), and then substituting in equation (30). The results, for both cases I and O are given as

$$Nu_I = Nu_O = 2.$$

FULLY DEVELOPED NATURAL CONVECTION IN AN OPEN-ENDED VERTICAL POROUS TUBE

Fundamental solutions for fully developed natural convection in a vertical porous tube can be considered as a special case of that obtained for the annuli, but with $N = 0$. However, two fundamental cases are only possible in the case of tube. These are:

fundamental solution of the second kind, case O

$$\left.\frac{\partial T}{\partial R}\right|_{R=0} = 0 \quad \left.\frac{\partial T}{\partial R}\right|_{R=1} = 1$$

fundamental solution of the third kind, case O

$$\left.\frac{\partial T}{\partial R}\right|_{R=0} = 0 \quad T(1,Z) = 1.$$

Solutions for the above two cases are obtained from that found for the annular geometry (cases O of the second and third kinds), but with $N = 0$.

RESULTS

To verify the validity of the analytical solutions obtained in the present work, these results are compared with the results of a similar problem solved for non-porous domains [10]. The comparison is plotted in Fig. 2 for the axial velocity radial distribution of the first kind fundamental solution.

Also, a sample of the results, showing both temperature and axial velocity distributions in the radial direction, is plotted in Figs. 3–6. This sample represents fundamental solutions for both first and fourth kinds and for both cases I and O.

CONCLUSIONS

Analytical solutions for fully developed upward (heating) or downward (cooling) natural convection velocity and temperature profiles in open-ended vertical concentric porous annuli have been obtained. These solutions correspond to four fundamental boundary conditions obtained by combining each of the two conditions of having one boundary maintained at UHF or at UWT with each of the conditions that the opposite boundary is kept adiabatic or isothermal at the inlet fluid temperature. Expressions for the fully developed volumetric flow rate, mixing cup temperature and local Nusselt number are presented for each considered case. Such fully developed values are approached, in a given annulus, when the height to gap width ratio (l/b) is sufficiently large. These values represent the limiting conditions and provide analytical checks on numerical solutions for transient developing flows.

Once a developing natural convection flow reaches a state of full development in a given annulus, the volumetric flow rate reaches its upper value; any further increase in the annulus height would not produce an increase in the volumetric flow rate. Moreover, for cases with an isothermal boundary in a given annulus, the Nusselt number reaches its lower limiting value while the mixing cup temperature reaches its upper limiting value and all remain constant, irrespective of any further increase in the channel height. However, for cases with two UHF boundary conditions, in a given annulus, the wall temperature (T_w) and the mixing cup temperature (T_m) continue their linear variations with further increases in the channel height.

REFERENCES

1. V. Prasad and F. A. Kulacki, Natural convection in a vertical porous annulus, *Int. J. Heat Mass Transfer* **27,** 207–219 (1984).
2. V. Prasad, F. A. Kulacki and M. Keyhani, Natural convection in porous media, *J. Fluid Mech.* **150,** 89–119 (1985).
3. R. Clarksean, N. Kwendakwema and R. Boehm, A study of mixed convection in a porous annulus between vertical concentric cylinders, *Proceedings of ASME/AIChE National Heat Transfer Conference*, Vol. 2, pp. 339–344. ASME, New York (1988).
4. D. C. Reda, Mixed convection in a liquid-saturated porous medium, *ASME J. Heat Transfer* **110,** 147–154 (1988).
5. K. Muralidhar, Mixed convection flow in a saturated porous annulus, *Int. J. Heat Mass Transfer* **32,** 881–888 (1989).
6. C. Y. Choi and F. A. Kulacki, Mixed convection through vertical porous annuli locally heated from the inner cylinder, *ASME J. Heat Transfer* **114,** 143–151 (1992).
7. C. Y. Choi and F. A. Kulacki, Non-Darcian effects on mixed convection in a vertical packed-sphere annulus, *ASME J. Heat Transfer* **115,** 506–510 (1993).
8. M. Parang and M. Keyhani, Boundary effects in laminar mixed convection flow through an annular porous medium, *ASME J. Heat Transfer* **109,** 1039–1041 (1987).
9. H. S. Kou and K. T. Lu, Combined boundary and inertia effects for fully developed mixed convection in a vertical channel embedded in porous media, *Int. Commun. Heat Mass Transfer* **20,** 333–345 (1993).
10. M. A. I. El-Shaarawi and M. A. Al-Nimr, Fully developed laminar natural convection in open-ended vertical concentric annuli, *Int. J. Heat Mass Transfer* **33,** 1873–1884 (1990).
11. M. A. Al-Nimr, Analytical solution for transient laminar fully developed free convection in vertical concentric annuli, *Int. J. Heat Mass Transfer* **36,** 2385–2395 (1993).
12. W. C. Reynolds, R. E. Lundberg and P. A. McCuen, Heat transfer in annular passages. General formulation of the problem for arbitrarily prescribed wall temperatures or heat fluxes, *Int J. Heat Mass Transfer* **6,** 483–493 (1963).